The extreme right-hand member is, however, the logarithmic derivative of $f(x)/F(x)$; hence by the preceding theorem, $f(x)/F(x)$ is constant, or $f(x) = k \cdot F(x)$: *If two functions have the same logarithmic derivative, their quotient is a constant*; or *If the logarithmic derivative of a function is given, that function is determined except for a constant factor.*

These are the fundamental theorems concerning logarithmic derivatives; they are precisely analogous to the corresponding theorems for ordinary derivatives.

5. Discontinuity. It has been shown above that $f'(x)$ exists whenever the relative rate r_r defined by (5) exists, provided only that $f(x)$ is continuous. It is, however, possible that $f'(x)$ and therefore also r_r is discontinuous.

It might appear that (8) precludes such a possibility, for the left side approaches r_r, and c approaches a as Δx approaches zero. But c does not necessarily take on all values; hence the approach of c to a may be only through a set of special values of x.

That the possibility just mentioned actually does occur is manifested by the example

$$f(x) = 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x} + 1, \qquad x \neq 0; \qquad f(0) = 1;$$

which has an indefinite integral:

$$\phi(x) = x^4 \sin\frac{1}{x} + x, \qquad x \neq 0; \qquad \phi(0) = 0.$$

Its derivative is

$$f'(x) = 12x^2 \sin\frac{1}{x} - 5x \cos\frac{1}{x} + \sin\frac{1}{x}, \qquad x \neq 0; \qquad f'(0) = 0;$$

which is discontinuous. Its logarithmic derivative may be derived either by (2) or by (5), and its value when $x = 0$ is zero.

6. Conclusion. The facts and illustrations given above parallel completely the fundamental theorems and illustrations usually given for ordinary derivatives. In deducing them, although the name *logarithmic* derivative has been used, the notion of logarithms has not been employed, and it has been shown that the ideas themselves are independent of the concept of logarithms.

TWO GEOMETRICAL APPLICATIONS OF THE METHOD OF LEAST SQUARES.

By J. L. COOLIDGE, Harvard University.

The two problems discussed in the present paper are taken from a recent work by Vahlen entitled "Konstruktionen und Approximationen."* It seems to me that they are sufficiently interesting and important to merit special attention.

* Leipzig, 1911, pages 125, 126.

Problem 1. *n lines are drawn in a plane which should be concurrent, but are not. If the lines be equally trustworthy, what point should be taken for their expected point of concurrence?*

The solution given (without proof) for this problem is that of Bertot.† It seems rather artificial, and it occurred to me that a more direct solution would have a certain interest. We wish to find such a point that the sum of the squares of its distances from n given lines shall be a minimum. The lines being given in normal form, we have

$$\sum_{i=1}^{i=n} (x \cos \alpha_i + y \sin \alpha_i - p_i)^2 = \text{Minimum}.$$

Equating to zero the partial derivatives as to x and y,

$$\begin{aligned} \sum_{i=1}^{i=n} x \cos^2 \alpha_i + \sum_{i=1}^{i=n} y \cos \alpha_i \sin \alpha_i &= \sum_{i=1}^{i=n} p_i \cos \alpha_i, \\ \sum_{i=1}^{i=n} x \cos \alpha_i \sin \alpha_i + \sum_{i=1}^{i=n} y \sin^2 \alpha_i &= \sum_{i=1}^{i=n} p_i \sin \alpha_i. \end{aligned} \tag{1}$$

Let us suppose that a unit circle has been drawn about the origin as center; the points (1, 0) and (0, 1) shall be called A and B respectively. Circles are drawn on OA and OB as diameters. A perpendicular from the origin on the ith line shall meet that line in P_i, while it meets the OA and OB circles in Q_i and R_i respectively. See Fig. 1. We have for the coördinates of these various points

$$P_i = (p_i \cos \alpha_i,\ p_i \sin \alpha_i), \quad Q_i = (\cos^2 \alpha_i,\ \cos \alpha_i \sin \alpha_i),$$
$$R_i = (\cos \alpha_i \sin \alpha_i,\ \sin^2 \alpha_i).$$

The centre of gravity P of the n points P_i is easily constructed, as are Q, R the centres of gravity of Q_i and R_i. Let their coördinates be

$$P = (\bar{x}, \bar{y}), \quad Q = (x_1, y_1), \quad R = (x_2, y_2), \quad y_1 = x_2.$$

Equations (1) will take the form

$$x_1 x + y_1 y = \bar{x}, \quad x_2 x + y_2 y = \bar{y},$$

$$x = \frac{\begin{vmatrix} \bar{x} & x_2 \\ \bar{y} & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} = \frac{\Delta OPR}{\Delta OQR} = \frac{\text{Distance } P - OR}{\text{Distance } Q - OR},$$

$$y = \frac{\begin{vmatrix} x_1 & \bar{x} \\ y_1 & \bar{y} \end{vmatrix}}{\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}} = \frac{\Delta OQP}{\Delta OQR} = \frac{\text{Distance } P - OQ}{\text{Distance } R - OQ}.$$

† "Solution géométrique du problème de la détermination du lieu le plus probable du navire," *Comptes Rendus*, Vol. LXXXII, 1876, p. 682.

Since we know the unit length already, the values x and y are easily found, and the required point determined.

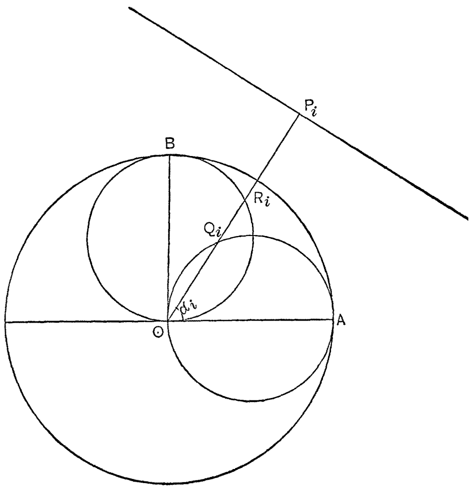

Fig. 1.

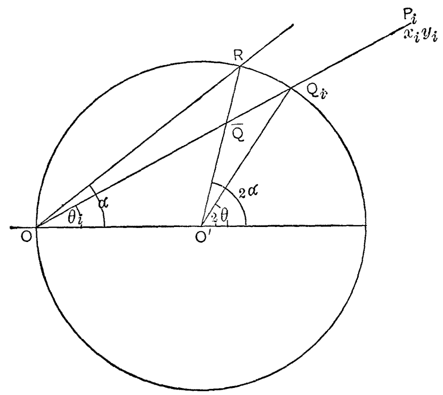

Fig. 2.

Problem 2. *n points are given in the plane which should be collinear, but are not. If the points be equally trustworthy, what line should be taken as their supposed common line?*

Vahlen's remark on this problem is the following:* "Für die konstruktive Ausgleichung einer Geraden aus mehr als zwei Punkten . . . scheinen dem Bertotschen entsprechende einfache Verfahren noch nicht gefunden zu sein. Sie wurden, wenn auch theoretisch von Interesse, doch praktisch schon zu kompliziert sein, um wirklich angewendet werden zu können." Without attempting to settle the obscure question of how complicated a construction may be without losing its practical importance, let us show that this problem is really easier of solution than the other. The given points shall be $(x_1, y_1) \cdots (x_n, y_n)$. We seek a line, the sum of the squares of whose distances from these shall be a minimum. If this line be given in normal form, we have

$$\sum_{i=1}^{i=n} (x_i \cos \alpha + y_i \sin \alpha - p)^2 = \text{Minimum}.$$

Equating to zero the partial derivative as to p, and dividing by n,

$$\frac{\sum_{i=1}^{i=n} x_i}{n} \cos \alpha + \frac{\sum_{i=1}^{i=n} y_i}{n} \sin \alpha - p = 0.$$

This shows that the line must at any rate pass through the center of gravity of all the points. Let us imagine that the origin has been moved to that point.

* Loc. cit., p. 126.

$$\sum_{i=1}^{i=n} (x_i \cos\alpha + y_i \sin\alpha)^2 = \text{Minimum}.$$

Equating to zero the derivative as to α,

$$\sum_{i=1}^{i=n} (y_i^2 - x_i^2) \sin\alpha \cos\alpha = \sum_{i=1}^{i=n} x_i y_i (\sin^2\alpha - \cos^2\alpha).$$

Let

$$\frac{y_i}{x_i} = \theta_i,$$

$$\tan 2\alpha = \frac{\sum_{i=1}^{i=n} \sin 2\theta_i}{\sum_{i=1}^{i=n} \cos 2\theta_i}.$$

The origin shall be O, the point (1, 0) shall be O'. We construct a unit circle about O' as center. Let OP_i meet this circle again in Q_i. The coördinates of Q_i in the system *where O' is origin* will be $(\cos 2\theta_i, \sin 2\theta_i)$. Let $\bar{Q}$ be the center of gravity of the points Q_i. The line $O'\bar{Q}$ shall meet the circle in R, that is to say, R is the intersection which lies on the same side of the diameter OO' as does $\bar{Q}$. Then OR is the required line. See Fig. 2.

COMPUTATION FORMULA FOR THE PROBABILITY OF AN EVENT HAPPENING AT LEAST C TIMES IN N TRIALS.

By E. C. MOLINA, New York City.

Many problems in biometry, radioactivity, etc., require for their quantitative solution a convenient formula for computing the probability of an event happening at least c times in n trials; the probability p of its happening in one trial being known. Several relatively simple formulas are given by Poisson in the third chapter of his *Recherches sur la Probabilité des Jugements*, availing himself of a method developed by Laplace in the *Théorie Analytique des Probabilités*. But these formulas explicitly exclude the case where p is very small and $c = np + r$, r not being small compared with np. For this range of values a problem connected with my engineering work forced me early in 1908 to develop the formulas* (5) and (6) given below.

Let P = the required probability, $\alpha = np$, $s = (n - c)/(c + 1)$,

$$F(x) = \frac{2c(1-x)x^2 + (n-s)sx^4}{[c-(n-s)x]^4}$$

* These formulas have been used for the construction of curves which are in constant use in the Engineering Department of the American Telephone and Telegraph Company. They are published in the hope that others will find them helpful.